Assignment 7.

This homework is due *Friday* March 21.

There are total 36 points in this assignment. 32 points is considered 100%. If you go over 32 points, you will get over 100% for this homework (but not over 115%) and it will count towards your course grade.

Collaboration is welcome. If you do collaborate, make sure to write/type your own paper *and give credit to your collaborators in your pledge.* Your solutions should contain full proofs. Bare answers will not earn you much.

This assignment is longer than usual, so it is worth $\mathbf{\frac{3}{2}}$ as much as a regular homework in terms of course grade.

1. Wilson's theorem

(1) (a) [2pt] (5.3.1a) Find the remainder of 15! when divided by 17.
   (b) [2pt] Find the remainder of 58! when divided by 61. (*Hint:* $59 \equiv -2 \pmod{61}$.)

(2) [2pt] (5.3.3) Arrange the integers $2, 3, 4, \ldots, 21$ in pairs a, b that satisfy $ab \equiv 1 \pmod{23}$.

(3) [2pt] (5.3.9) Using Wilson's theorem, prove that for any odd prime p,
$$1^2 \cdot 3^2 \cdot 5^2 \cdots (p-2)^2 \equiv (-1)^{(p+1)/2} \pmod{p}.$$
(*Hint:* Using that $k \equiv -(p-k) \pmod{p}$, show that
$$2 \cdot 4 \cdot 6 \cdots (p-1) \equiv (-1)^{(p-1)/2} 1 \cdot 3 \cdot 5 \cdots (p-2) \pmod{p}.)$$

(4) [3pt] (5.3.18) Prove that if p and $p+2$ are a pair of twin primes, then
$$4((p-1)! + 1) + p \equiv 0 \pmod{p(p+2)}.$$

2. Number-theoretic functions

(5) [2pt] (6.1.6) For any integer $n \geq 1$, establish that $\tau(n) \leq 2\sqrt{n}$. (*Hint:* In each pair s.t. $d_1 d_2 = n$, at least one of numbers is $\leq \sqrt{n}$.)

(6) (6.1.7) Prove the following.
   (a) [2pt] $\tau(n)$ is an odd integer if and only if n is a perfect square.
   (b) [2pt] $\sigma(n)$ is an odd integer if and only if n is of the form m^2 or $2m^2$. (*Hint:* If p is an odd prime, then $1 + p + \ldots + p^k$ is odd if and only if k is even.)

(7) (6.1.14) For $k \geq 2$, show each of the following:
   (a) [1pt] $n = 2^{k-1}$ satisfies the equation $\sigma(n) = 2n - 1$.
   (b) [1pt] If $2^k - 1$ is prime, then $n = 2^{k-1}(2^k - 1)$ satisfies the equation $\sigma(n) = 2n$.
   (c) [1pt] If $2^k - 3$ is prime, then $n = 2^{k-1}(2^k - 3)$ satisfies the equation $\sigma(n) = 2n + 2$.

   Comment. It is an open question if there are any positive integers such that $\sigma(n) = 2n + 1$.

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(8) [3pt] Numbers with the property $\sigma(n) = 2n$ are called *perfect numbers*. In other words, perfect numbers are those equal to sum of their divisors, excluding the number itself. For example, $28 = 1 + 2 + 4 + 7 + 14$. In problem 7b we proved that numbers of the form $n = 2^{k-1}(2^k - 1)$, with $2^k - 1$ prime, are perfect. Prove the partial converse: if a perfect number is *even*, then n has the form $n = 2^{k-1}(2^k - 1)$, where $2^k - 1$ is prime. (*Hint:* Represent n as $2^k \cdot m$, where m is odd. Use multiplicativity of σ.)

Comment. The statement above says nothing about odd perfect numbers, and there is a good reason: it is an open question whether they exist.

3. Muliplicativity

(9) [2pt] (6.1.18) Let f, g be multiplicative functions that have the property $f(p^k) = g(p^k)$ for each prime p and $k \in \mathbb{Z}$, $k \geq 0$. Prove that $f = g$.

(10) (6.1.20) Let $\omega(n)$ be the number of *distinct prime* divisors of $n > 1$, with $\omega(1) = 0$. For instance, $\omega(360) = \omega(2^3 \cdot 3^2 \cdot 5) = 3$.
  (a) [1pt] Show that $2^{\omega(n)}$ is a multiplicative function.
  (b) [3pt] For a positive integer n, establish the formula
$$\tau(n^2) = \sum_{d|n} 2^{\omega(d)}.$$
(*Hint:* Argue that both LHS and RHS are multiplicative functions. Use Problem 9.)

(11) ($\sim$6.1.22) τ and σ are particular cases of a family of number theoretic functions σ_s, $s \in \mathbb{R}$:
$$\sigma_s(n) = \sum_{d|n} d^s,$$
so $\tau = \sigma_0$ and $\sigma = \sigma_1$.
  (a) [2pt] (6.1.8) Prove that $\sigma_{-1}(n) = \sigma(n)/n$. (*Hint:* Multiply both sides by n.)
  (b) [1pt] (6.1.17) Show that for every fixed $s \in \mathbb{R}$, function n^s is multiplicative.
  (c) [2pt] Prove that σ_s is a multiplicative function for every $s \in \mathbb{R}$. (*Hint:* Use item 11b.)
  (d) [2pt] Let $s \neq 0$. If $n = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}$ is the prime factorization of n, then
$$\sigma_s(n) = \left(\frac{p_1^{s(k_1+1)} - 1}{p_1^s - 1} \right) \left(\frac{p_2^{s(k_2+1)} - 1}{p_2^s - 1} \right) \cdots \left(\frac{p_r^{s(k_r+1)} - 1}{p_r^s - 1} \right).$$
(*Hint:* Compute $\sigma_s(n)$ in the case $n = p^k$. Then use multiplicativity.)